Signatures of hermitian forms and applications

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(joint work with Vincent Astier)

In [1], [2] and [3] we started developing the theory of signatures of hermitian forms, defined over central simple algebras with involution (with respect to orderings on the base field), inspired by [4]. In contrast to classical signatures of quadratic forms, signatures of hermitian forms should be considered as relative invariants. Below we present a summary of our work thus far.

Let F be a formally real field with space of orderings X_F and Witt ring $W(F)$. Let (A,σ) be an F-algebra with involution, i.e. a pair consisting of a finite-dimensional F-algebra A, whose centre $Z(A)$ satisfies $[Z(A):F]\leq 2$, and which is assumed to be either simple (if $Z(A)$ is a field) or a direct product of two simple algebras (if $Z(A)=F\times F$), and an F-linear involution $\sigma: A\to A$. For $\varepsilon\in\{-1,1\}$ let $W_\varepsilon(A,\sigma)$ be the Witt group of Witt equivalence classes of ε-hermitian forms defined on finitely generated right A-modules. This is a $W(F)$-module. All forms are assumed to be non-singular and are identified with their classes in $W_\varepsilon(A,\sigma)$.

Let A be Brauer equivalent to an F-division algebra D and let ϑ be an involution on D of the same kind as σ. Then (A,σ) and (D,ϑ) are Morita equivalent and we obtain a (non-canonical) isomorphism of $W(F)$-modules $W_\varepsilon(A,\sigma)\simeq W_{\varepsilon\mu}(D,\vartheta)$ with $\mu\in\{-1,1\}$. For the purpose of the study of signatures we may assume that $\varepsilon=\mu=1$, cf. [1, 2.1].

Let $P\in X_F$, let F_P denote a real closure of F at P and consider

$$(\star)\qquad W(A,\sigma)\longrightarrow W(A\otimes_F F_P,\sigma\otimes\mathrm{id})\xrightarrow{\mathscr{M}_P} W_\varepsilon(D_P,\vartheta_P)\xrightarrow{\mathrm{sign}_P}\mathbb{Z},$$

where the first map is induced by scalar extension, the second map is an isomorphism of $W(F_P)$-modules induced by Morita equivalence and sign_P is either the classical signature isomorphism if $\varepsilon=1$ and $(D_P,\vartheta_P)\in\{(F_P,\mathrm{id}),(F_P(\sqrt{-1}),-),((-1,-1)_{F_P},-)\}$ (where $-$ denotes conjugation and quaternion conjugation, respectively), or $\varepsilon=-1$ and $\mathrm{sign}_P\equiv 0$ if $(D_P,\vartheta_P)\in\{(F_P,\mathrm{id}),((-1,-1)_{F_P},-),(F_P\times F_P,\widehat{\ })\}$ (where $\widehat{\ }$ denotes the exchange involution), and where in each case the indicated involutions are obtained after a further application of Morita equivalence. We call $\mathrm{Nil}[A,\sigma]:=\{P\in X_F\mid \mathrm{sign}_P\equiv 0\}$ the set of *nil-orderings* of (A,σ). It depends only on the Brauer class of A and the type of σ. In addition it is clopen in X_F [1, 6.5]. We write $\widetilde{X}_F:=X_F\setminus\mathrm{Nil}[A,\sigma]$.

Definition 1. Let $h\in W(A,\sigma)$, $P\in X_F$ and $\mathscr{M}_P$ as in $(\star)$. The *M-signature* of h at $(P,\mathscr{M}_P)$ is defined by $\mathrm{sign}_P^{\mathscr{M}_P} h:=\mathrm{sign}_P(\mathscr{M}_P(h\otimes F_P))$ and is independent of the choice of F_P.

If we choose a different Morita map $\mathscr{M}_P'$ in $(\star)$, then $\mathrm{sign}_P^{\mathscr{M}_P'} h=\pm\,\mathrm{sign}_P^{\mathscr{M}_P} h$, cf. [1, 3.4], which prompts the question if there is a way to make the M-signature independent of the choice of Morita equivalence. It follows from [1, 6.4] and [2, 3.2] that:

Theorem 2. *There exists* $H \in W(A,\sigma)$ *such that* $\operatorname{sign}_P^{\mathscr{M}_P} H \neq 0$ *for all* $P \in \widetilde{X}_F$.

Definition 3. Let $P \in \widetilde{X}_F$, let $\mathscr{M}_P$ be any Morita map as in ($\star$), let H be as in (2) and let $\delta \in \{-1,1\}$ be the sign of $\operatorname{sign}_P^{\mathscr{M}_P} H$. Let $h \in W(A,\sigma)$. The *H-signature of h at P* is defined by $\operatorname{sign}_P^H h := \delta \operatorname{sign}_P^{\mathscr{M}_P} h$. If $P \in \operatorname{Nil}[A,\sigma]$, we set $\operatorname{sign}_P^H h := 0$.

The H-signature at P is independent of the choice of Morita equivalence $\mathscr{M}_P$ and is a refinement of the definition of signature in [4], the latter not being defined when σ becomes hyperbolic over $A \otimes_F F_P$, cf. [1, 3.11]. The H-signature has many pleasing properties, cf. [5, 4.1] for (*iv*) and [1, 3.6, 8.1] for the other statements:

Theorem 4.

(*i*) *Let h be a hyperbolic form over (A,σ), then* $\operatorname{sign}_P^H h = 0$.

(*ii*) *Let* $h_1, h_2 \in W(A,\sigma)$, *then* $\operatorname{sign}_P^H(h_1 \perp h_2) = \operatorname{sign}_P^H h_1 + \operatorname{sign}_P^H h_2$.

(*iii*) *Let* $h \in W(A,\sigma)$ *and* $q \in W(F)$, *then* $\operatorname{sign}_P^H(q \cdot h) = \operatorname{sign}_P q \cdot \operatorname{sign}_P^H h$.

(*iv*) (*Pfister's local-global principle*) *Let* $h \in W(A,\sigma)$. *Then h is a torsion form if and only if* $\operatorname{sign}_P^H h = 0$ *for all* $P \in X_F$.

(*v*) (*Going-up*) *Let* $h \in W(A,\sigma)$ *and let L/F be an algebraic extension of ordered fields. Then* $\operatorname{sign}_Q^{H\otimes L}(h \otimes L) = \operatorname{sign}_{Q\cap F}^H h$ *for all* $Q \in X_L$.

(*vi*) (*Going-down: Knebusch trace formula*) *Let L/F be a finite extension of ordered fields and let* $h \in W(A \otimes_F L, \sigma \otimes \mathrm{id})$. *Then* $\operatorname{sign}_P^H(\operatorname{Tr}^*_{A\otimes_F L} h) = \sum_{P\subseteq Q\in X_L} \operatorname{sign}_Q^{H\otimes L} h$ *for all* $P \in X_F$, *where* $\operatorname{Tr}^*_{A\otimes_F L} h$ *denotes the Scharlau transfer induced by the A-linear homomorphism* $\mathrm{id}_A \otimes \operatorname{Tr}_{L/F} : A \otimes_F L \to A$.

The pair $(\ker \operatorname{sign}_P, \ker \operatorname{sign}_P^H)$ is a prime m-ideal of the $W(F)$-module $W(A,\sigma)$ whenever $P \in \widetilde{X}_F$ in the following sense, cf. [2, 4.1]:

Definition 5. Let R be a commutative ring and let M be an R-module. An *m-ideal* of M is a pair (I,N) where I is an ideal of R, N is a submodule of M, and such that $I \cdot M \subseteq N$.

An m-ideal (I,N) of M is *prime* if I is a prime ideal of R (we assume that all prime ideals are proper), N is a proper submodule of M, and for every $r \in R$ and $m \in M$, $r \cdot m \in N$ implies that $r \in I$ or $m \in N$.

We obtain a classification à la Harrison and Lorenz-Leicht, cf. [2, 5.5, 5.7]:

Theorem 6. *Let (I,N) be a prime m-ideal of the $W(F)$-module $W(A,\sigma)$.*

(*a*) *If* $2 \notin I$, *then one of the following holds:*

  (*i*) *There exists* $P \in X_F$ *such that* $(I,N) = (\ker \operatorname{sign}_P, \ker \operatorname{sign}_P^H)$.

  (*ii*) *There exist* $P \in X_F$ *and a prime* $p > 2$ *such that* $(I,N) = \big(\ker(\pi_p \circ \operatorname{sign}_P), \ker(\pi \circ \operatorname{sign}_P^H)\big)$, *where* $\pi_p : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ *and* $\pi : \operatorname{Im}\operatorname{sign}_P^H \to \operatorname{Im}\operatorname{sign}_P^H / (p \cdot \operatorname{Im}\operatorname{sign}_P^H)$ *are the canonical projections.*

(*b*) *If* $2 \in I$, *then* $I = I(F)$, *the fundamental ideal of* $W(F)$. *Furthermore, a pair* $(I(F),N)$ *is a prime m-ideal of* $W(A,\sigma)$ *if and only if N is a proper submodule of* $W(A,\sigma)$ *with* $I(F) \cdot W(A,\sigma) \subseteq N$.

When $2 \in I$, N is not uniquely determined by I (in contrast to the $2 \notin I$ case), since there are in general several proper submodules N of $W(A,\sigma)$ containing $I(F)\cdot W(A,\sigma)$, such as $I(F)\cdot W(A,\sigma)$ itself and $I(A,\sigma)$, the submodule of $W(A,\sigma)$ consisting of all classes of forms of even rank. In general $I(F)\cdot W(A,\sigma) \neq I(A,\sigma)$, cf. [2, 5.8]. Also, $I(A,\sigma)$ can be singled out by a natural property, cf. [2, 5.10].

The following result is [1, 7.2]:

Theorem 7. *Let $h \in W(A,\sigma)$. The total H-signature $\operatorname{sign}^H h : X_F \to \mathbb{Z}, P \mapsto \operatorname{sign}^H_P h$ is continuous (with respect to the Harrison topology on X_F and the discrete topology on $\mathbb{Z}$).*

Finally, we present some results from [3]. Let $C(X_F,\mathbb{Z})$ denote the ring of continuous functions from X_F to $\mathbb{Z}$ and consider the group homomorphism $\operatorname{sign}^H : W(A,\sigma) \to C(X_F,\mathbb{Z}), h \mapsto \operatorname{sign}^H h$.

Theorem 8. *For every $f \in C(X_F,\mathbb{Z})$ there exists $n \in \mathbb{N}$ such that $2^n f \in \operatorname{Im}\operatorname{sign}^H$. In other words, the cokernel of sign^H is a 2-primary torsion group.*

Definition 9. The *stability index* of (A,σ) is the smallest $k \in \mathbb{N}$ such that $2^k C(X_F,\mathbb{Z}) \subseteq \operatorname{Im}\operatorname{sign}^H$ if such a k exists and ∞ otherwise. It is independent of the choice of H. The group $\operatorname{coker}\operatorname{sign}^H$ is up to isomorphism independent of the choice of H. We denote it by $S_H(A,\sigma)$ and call it the *stability group* of (A,σ).

It follows from Theorems 4(*iv*) and 8 and from [6, 6.1] that

Theorem 10. *Let $W_t(A,\sigma)$ denote the torsion subgroup of $W(A,\sigma)$. The sequence*

$$0 \longrightarrow W_t(A,\sigma) \longrightarrow W(A,\sigma) \xrightarrow{\operatorname{sign}^H} C(X_F,\mathbb{Z}) \longrightarrow S_H(A,\sigma) \longrightarrow 0$$

is exact. The groups $W_t(A,\sigma)$ and $S_H(A,\sigma)$ are 2-primary torsion groups.

References

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